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TRANSFORMATION GROUPS ON RIEMANNIAN SYMMETRIC SPACES

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1. On any Riemannian space N, we denote by $I(N)$ (resp, by $I(N)^0$) the group of all isometries (resp. the identity connected component of $I(N)$). The purpose of this note is to prove the following results.

Theorem 1. *Let M be a Riemannian symmetric space of the noncompact type, and L a (not necessary connected) effective Lie transformation group on M. If $L \supset I(M)^0$, then $I(M) \supset L$.*

Corollary (*E. Cartan*). *Let M be a Riemannian symmetric space of the noncompact type. Then $I(M)$ is isomorphic to the group of all automorphisms of $I(M)^0$.*

Theorem 2. *Let M be an irreducible Riemannian symmetric space which is not of the Euclidian type, and let $\mathscr{D}$ be an $I(M)^0$-invariant differential operator on M. Then any transformation f of M leaving $\mathscr{D}$ invariant is an isometry.*

Theorem 1 has been proved partially in [4]. The author wishes to thank Professor T. Nagano for his valuable and generous suggestions.

2. Denote by G_n the isotropy subgroup of any transitive transformation group G on a manifold N at any point $n \in N$, and by M a Riemannian symmetric space of the noncompact type unless stated otherwise. Then the following properties of M are well known:

(i) *M is homeomorphic to an open cell.*

(ii) *Each irreducible factor in the de Rham decomposition of M is again a Riemannian symmetric space of the noncompact type.*

(iii) *$I(M)^0$ is semi-simple.*

(iv) *$I(M)^0$ is also the identity connected component of the group of all isometries of any $I(M)^0$-invariant metric on M.*

(v) *$I(M)^0_m$ $(m \in M)$ is a maximal compact subgroup of $I(M)^0$.*

(vi) *$I(M)^0_m \neq I(M)^0_n$ if $m \neq n$ $(m, n \in M)$.*

A Riemannian symmetric space N (not of the Euclidian type) is irreducible if and only if $I(N)^0$ is simple, and in this case, the linear isotropy representation of $I(N)^0_n$ $(n \in N)$ is irreducible.

Lemma 1. *Let G be an effective Lie transformation group on M such that*

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$G \supset I(M)^0$. If an element $g \in G$ commutes with each element in $I(M)^0$, then g is the identity transformation. In particular, the center of G consists of the identity transformation.

Proof. Let m be a point in M. Since $hg(m) = gh(m) = g(m)$ for any $h \in I(M)^0_m$ and $g(m) = h'g(m) = gh'(m)$ for any $h' \in I(M)^0_{g(m)}$, we have $I(M)^0_m = I(M)^0_{g(m)}$, and hence $g(m) = m$ by (vi). Thus g is the identity.

Lemma 2. *If G is a connected, transitive Lie transformation group on M, then, for any $m \in M$,*

(a) *G_m is connected,*

(b) *a maximal compact subgroup of G_m is also a maximal compact subgroup in G.*

Proof. (a) This is due to that M is simply connected. (b) Let H (resp. H') be a maximal compact subgroup of G_m (resp. G) such that $H' \supset H$. Then G/H' and G_m/H are homeomorphic to an open cell. Consider the canonical fibration: $G/H \to G/G_m$ whose standard fibre is G_m/H. Since $G/G_m (\cong M)$ and G_m/H are homeomorphic to an open cell, G/H is also homeomorphic to an open cell. Consider another canonical fibration: $G/H \to G/H'$ whose standard fibre is H'/H. Since G/H and G/H' are homeomorphic to an open cell, H'/H must be contractible. Hence $H = H'$.

Lemma 3. *Let G be a connected, effective Lie transformation group on M such that $G \supset I(M)^0$. Then any connected abelian normal subgroup A of G cannot be transitive on M.*

Proof. Suppose that A were transitive. Since A is effective, $A_m (m \in M)$ is the identity. Thus A acts simply transitively on M. In particular A is a vector group. We fix a $m \in M$, and let φ: $A \to M$ be the natural identification map of A with M defined by $\varphi(a) = a(m)$ $(a \in A)$. For each $g \in I(M)^0$, we define a transformation $g^*: A \to A$ such that $\varphi \circ g^* = g \circ \varphi$. Then g^* is an affine transformation of A. In fact, $g^*(a) = \varphi^{-1} \cdot g \cdot \varphi(a) = \varphi^{-1}(ga(m)) = \varphi^{-1}(gag^{-1} \cdot g(m)) = gag^{-1} + \sigma(g)$, where $\sigma(g) \in A$ is defined by $\sigma(g)(m) = g(m)$. Then it is easy to see that the map: $g \to g^*$ is a faithful representation of $I(M)^0$ into the affine group of A. Since any affine representation of semi-simple Lie groups has a fixed point, $I(M)^0$ has a fixed point in A. This is a contradiction.

Lemma 4. *Let G be as in Lemma 3. Then G is semi-simple.*

Proof. Let A be any connected abelian normal subgroup of G. It suffices to show that A is trivial. In fact, the orbits of A define $I(M)^0$ invariant foliation. Therefore each orbit is again a Riemannian symmetric space of the non-compact type (c.f. (ii)). Therefore, by Lemma 3, each orbit of A must be a point. Since G is effective, A must be the identity.

The following Lemma 5 is well-known.

Lemma 5. *Let G be any connected semi-simple Lie group without center. Then any connected subgroup of G which properly contains a maximal compact* *subgroup of G contains a normal subgroup of G.*

Lemma 6. *If G is a connected effective Lie transformation group on M such that $G \supset I(M)^0$, then $G = I(M)^0$.*

Proof. By Lemma 4, G is semi-simple. By Lemma 2, $G_m (m \in M)$ contains a maximal compact subgroup of G. Since the center of G is trivial by Lemma 1 and G is effective, G_m $(m \in M)$ is a maximal compact subgroup of G by Lemma 5. In particular, M has a G-invariant metric, which is, of course, $I(M)^0$-invariant. Hence $G = I(M)^0$ (*c.f.* (iv)).

Lemma 7. *Let N be a Riemannian symmetric space such that each irreducible factor of N is not of the Euclidian type. If G is an effective transformation group on N such that G contains $I(N)^0$ as a normal subgroup, then $G \subset I(N)$.*

Proof. First assume N is irreducible. Denote the metric of N by τ. For any $g \in G$, $g\tau$ is also an $I(M)^0$-invariant metric, since $l(g\tau) = g(g^{-1}lg\tau) = g\tau$ for any $l \in I(M)^0$. Since N is irreducible, $g\tau = c\tau$ for some constant real number c. In particular, g is a homothetic transformation. Since N is complete and not of the Euclidian type, g is an isometry. Thus $G \subset I(M)$ when N is irreducible. The general case where N is reducible can be easily verified if we note that g transforms each irreducible factor of N onto an irreducible factor.

Proof of Theorem 1. By Lemma 6, the identity connected component of L is equal to $I(M)^0$. Therefore L contains $I(M)^0$ as a normal subgroup. Thus by Lemma 7, $L \subset I(M)$.

3. Let G be a Lie group, and G^0 its identity connected component. For any $g \in G$, we denote by Inn (g) the inner automorphism of G^0 defined by g. We also denote by Ad (g) the automorphism of the Lie algebra of G^0 induced by Inn (g).

Proof of the Corollary. Define a natural homomorphism $\iota: I(M) \to$ Aut $(I(M)^0)$ by $\iota(g) =$ Inn (g). Then by Lemma 1, ι is injective. Therefore we can identify $I(M)^0$ with $\iota(I(M)^0)$. We define Aut $(I(M)^0)_m$ by $\{\varphi \in \text{Aut}\,(I(M)^0) \mid \varphi(I(M)^0_m) \subset I(M)^0_m\}$. Then by (vi), Aut $(I(M)^0)_m \cap I(M)^0 = I(M)^0_m$ by (vi), and Aut $(I(M)^0)/$Aut$(I(M)^0)_m$ is identified with $M = I(M)^0/I(M)^0_m$. Thus Aut $(I(M)^0)$ can be considered as a Lie transformation group on M which contains $I(M)^0$, and the corollary follows from Theorem A and the following lemma.

Lemma 8. Aut $(I(M)^0)$ *is effective on M.*

Proof. We denote by $\mathfrak{J}$ (resp. $\mathfrak{J}_m$) the Lie algebra of $I(M)^0$ (resp. $I(M)^0_m$), and also donote by $\mathfrak{p}_m$ the orthogonal complement of $\mathfrak{J}_m$ with respect to the Killing form of $\mathfrak{J}$. We remark that $[\mathfrak{p}_m, \mathfrak{p}_m] = \mathfrak{J}_m$. Since any automorphism of $\mathfrak{J}$ preserves the Killing form of $\mathfrak{J}$, Ad $(\varphi)(\mathfrak{p}_m) \subset \mathfrak{p}_m$ for any $\varphi \in$ Aut $(I(M)^0)_m$, and the linear isotropy representation of a $\varphi \in$ Aut $(I(M)^0)_m$ is exactly the restriction of Ad (φ) to $\mathfrak{p}_m$. Therefore if a $\varphi \in$ Aut $(I(M)^0)_m$ operates on M as the identity, then $\mathrm{Ad}(\varphi)$ is the identity on $\mathfrak{p}_m$. Since $[\mathfrak{p}_m, \mathfrak{p}_m] = \mathfrak{J}_m$, $\mathrm{Ad}(\varphi)$ is the identity on $\mathfrak{J}$. Hence φ is the identity, and $\mathrm{Aut}(I(M)^0)$ is effective.

4. Let V be a finite dimensional vector space over the field $\mathfrak{k}$. Denote by $S^k(V)$ the vector space of all k-th contravariant symmetric tensors of V. The group $GL(V)$ of all linear isomorphism of V naturally operates on $S^k(V)$. An element B of $S^k(V)$ is defined to be *non-degenerate* if there is no non-zero vector ζ in V^* (the dual vector space of V) such that $\iota(\zeta)B = 0$, where $\iota(\zeta)B$ denotes the usual inner product of B with ζ.

For any subspace $\mathfrak{g}$ of $V \otimes V^*$, we define $\mathfrak{g}^k$ by

$$\mathfrak{g}^k = \mathfrak{g} \otimes S^k(V^*) \cap V \otimes S^{k+1}(V^*) .$$

$\mathfrak{g}$ is defined to be of finite type if $\mathfrak{g}^k = 0$ for some k.

Theorem A (*Guillemin-Quillen-Sternberg* [2]). *Let $\mathfrak{k}$ be the field of complex numbers. Then a subspace $\mathfrak{g}$ of $V \otimes V^*$ is of finite type if and only if $\mathfrak{g}$ contains no element of rank* 1 (*i.e. an element of the form $v \otimes \zeta$, $v \in V$, $\zeta \in V^*$*).

Lemm 9. *Let $\mathfrak{k}$ be the field of real numbers, and G a Lie subgroup of $GL(V)$. If there is an element B in $S^k(V)$ $(k \geq 2)$ which is non-degenerate and invariant under G, then the Lie algebra $\mathfrak{g}(\subset V \otimes V^*)$ is of finite type.*

Proof. Denote the complexification of V (resp. B, $\mathfrak{g}$) by V_* (resp. B_*, $\mathfrak{g}_*$). Then $B_* \in S^k(V_*)$ is also non-degenerate. It suffices to show that $\mathfrak{g}_*$ is of finite type. In view of Theorem A, it suffices to show that g_* has no element of rank 1. Now suppose that g_* has an element T of rank 1. Then we can choose a basis $\{v_1, \cdots, v_n\}$ $(n = \dim V)$ such that $T(v_1) \neq 0$ and $T(v_i) = 0$ if $i \neq 1$. Since B_* is invariant under G, $\sum_{l=1}^{k} B_*(v_{j(1)}, \cdots, T(v_{j(l)}), \cdots, v_{j(k)}) = 0$. Therefore $\iota(T(v_1))B_* = 0$, which is impossible since B_* is non-degenerate.

For an n-dimensional smooth manifold N, we denote the frame bundle of N by $\mathscr{F}(N)$. If we fix an n-dimensional real vector space W, then $\mathscr{F}(N)$ is a principal $GL(W)$-bundle, and the fibre $\mathscr{F}(N)_n$ of $\mathscr{F}(N)$ over $n \in N$ can be considered as the totality of linear isomorphisms of W onto the tanget space $T(N)_n$ of N at n. The right operation of $GL(W)$ on $\mathscr{F}(N)$ is natural one, given by $\mathscr{F}(N)_n \times GL(W) \ni (x, a) \mapsto xa = x \cdot a \in \mathscr{F}(N)_n$ $(n \in N)$. For any diffeomorphism f of N, its differential df can be considered as a diffeomorphism of $\mathscr{F}(N)$. Let G be a Lie subgroup of $GL(W)$. A G-subbundle P of $\mathscr{F}(N)$ is called a *G-structure* on N. Note that P is a submanifold of $\mathscr{F}(N)$. A diffeomorphism f of N is called a *G-automorphism* if $(df)(P) \subset P$. A G-structure P is called to be of *finite type* if the Lie algebra g of $G(g \subset W \otimes W^*)$ is of finite type.

The following theorem is fundamental.

Theorem B [7], [6]. *Let P be a G-structure of finite type on N. Then the group* Aut (P) *of all G-automorphisms is a finite dimensional Lie transfor-*

mation group on N.

Lemma 10. *Let N be an irreducible Riemannian symmetric space, which is not of the Euclidian type, and $\mathscr{D}$ be an $I(N)^0$-invariant differential operator on N. Then the group L of all transformations of N, which leave $\mathscr{D}$ invariant, is a finite dimensional Lie transformation group.*

Proof. The homogeneous part of the highest degree of $\mathscr{D}$ defines a $I(N)^0$-invariant contravariant symmetric tensor $S(\mathscr{D})$ on N. Clearly $S(\mathscr{D})$ is nowhere zero. Fix a point $n \in N$, since the linear isotropy representation of $I(N)^0_n$ is irreducible, $S(\mathscr{D})_n$ is non-degenerate. In fact, the vector subspace of all elements ζ such that $\iota(\zeta)S(\mathscr{D})_n = 0$ is a $I(N)^0_n$-invariant subspace. Define a Lie subgroup $G(\mathscr{D})$ of $GL(T(N)_n)$ by $\{g \in GL(T(N)_n) \mid g(S(\mathscr{D})_n) = S(\mathscr{D})_n\}$. Since $S(\mathscr{D})$ is $I(N)^0$-invariant and $I(N)^0$ is transitive on N, $S(\mathscr{D})$ canonically defines a $G(\mathscr{D})$-structure $P(\mathscr{D})$ on N. In fact, the fibre of $P(\mathscr{D})$ over $n' \in N$ consists of all linear isomorphisms $x\colon T(N)_n \to T(N)_{n'}$ such that $x(S(\mathscr{D})_{n'}) = S(\mathscr{D})_n$. By Lemma 9, $P(\mathscr{D})$ is of finite type. Therefore, by Theorem B, Aut $(P(\mathscr{D}))$ is a finite dimensional Lie transformation group on N. Since $S(\mathscr{D})$ is also L-invariant, L is a subgroup of Aut $(P(\mathscr{D}))$. It is easy to see that L is closed in Aut $(P(\mathscr{D}))$. Therefore L is also a finite dimensional Lie transformation group.

Lemma 11. *With the same notation as in Lemma* 10, *the identity connected component L^0 of L is equal to $I(N)^0$.*

Proof. Since the lemma has already been proved for N of the noncompact type, we now consider the case where N is of the compact type. If L^0 is strictly greater than $I(N)^0$, then the linear isotropy representation of $I(N)^0_n$ ($n \in N$) contains a non-trivial scalar multiplication [5]. Therefore L^0_n cannot leave $S(\mathscr{D})_n$ invariant, and hence L^0 cannot be strictly greater than $I(N)^0$.

Proof of Theorem 2. Theorem 2 follows from Lemma 10 and Theorem 1 for N of the noncompact type, and from Lemma 11 and Lemma 7 for N of the compact type.

Appendix. Since [2] has not yet been published, we shall give a proof of Lemma 10 without using Theorem A. We use the same notation as in the proof of Lemma 10.

Another Proof of Lemma 10. We denote the Lie algebra of $G(\mathscr{D})$ by $\mathfrak{G}$, and have only to check the case where $\mathfrak{G}$ is of infinite type. Since $\mathfrak{G}$ contains the linear isotropy Lie algebra which is irreducible, so is $\mathfrak{G}$. By the classification theorem of irreducible infinite Lie algebras [4], we have two cases which may be possible:

(i) $\mathfrak{G}$ contains an element of rank 1,

(ii) $T(N)_n$ has a complex structure which $\mathfrak{G}$ leaves invariant.

Since $\mathfrak{G}$ leaves $S(\mathscr{D})_n$ invariant, $\mathfrak{G}$ contains no element of rank 1 (see the proof of Lemma 9). Therefore only the case (ii) might be possible. In this case our $G(D)$ structure defines a $I(N)^0$-invariant almost complex structure on N. It is well-known that an $I(N)^0$-invariant almost complex structure is unique on N

and integrable. Thus N is a Hermitian symmetric space and Aut $(P(\mathscr{D}))$ is a subgroup of the group of holomorphic transformations which is known to be a finite dimensional Lie group.

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